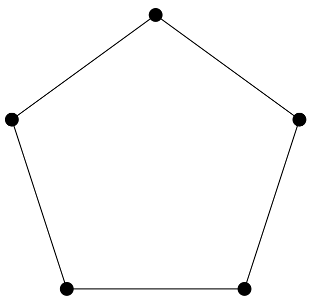
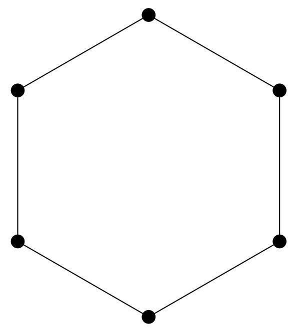
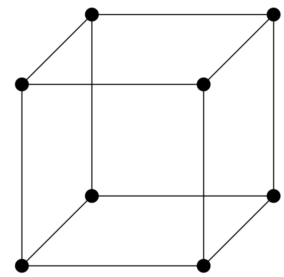
Chapter 5.4 - Edge Colorings

As one can color vertices, one can also color the edges of a graph. Here we require that any two edges that share a common end-vertex are colored differently. The smallest number of needed colors to color the edges of a graph G is called the *chromatic index* of G, and it is denoted by $\chi'(G)$. Note that each color class induces a matching of the graph.

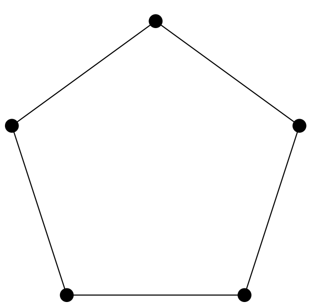
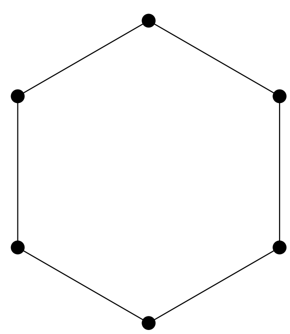
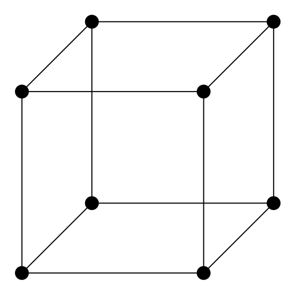
1: Find chromatic index of C_5, C_6, and the 3D-hypercube Q_3.

Solution: $\chi'(C_5) = 3$,
$\chi'(C_6) = 2$,
$\chi'(Q_3) = 3$,

An edge coloring of a graph G can be considered as a vertex coloring of its line graph $L(G)$. Recall that $V(L(G)) = E(G)$, and two vertices $e, f \in V(L(G))$ are adjacent when edges e in f are incident in G. So we have the following claim.

2: Find line graphs of C_5, C_6, and Q_3.

Solution: I hope you know how to do this...

Proposition 1. *For any graph G,*

$$\chi'(G) = \chi(L(G)).$$

3: Why is the proposition true?

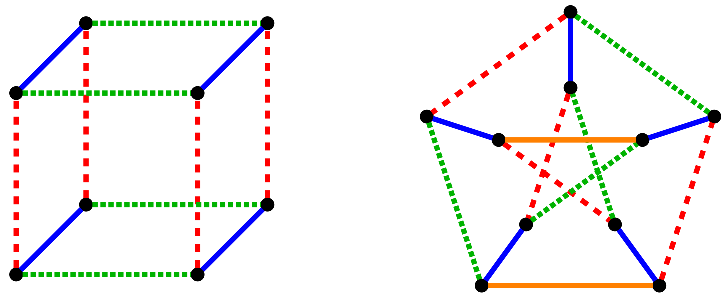
Solution: Coloring edges in G is the same as coloring vertices in $L(G)$.

Obviously, we need at least $\Delta(G)$ colors to color the edges of G, i.e., $\chi'(G) \geq \Delta(G)$. Surprisingly, $\Delta(G)+1$ will be always enough - Vizing's Theorem later.

Notice that Q_3 and C_6 are bipartite graph, and it has chromatic index as its maximum degree. With the following classical theorem of König from 1916, we will see that this is a case for every bipartite graph.

First, we do the following observation.

4: Let c be an edge-coloring of a graph G. Let α and β be two distinct colors. How does the subgraph of G induced by edges colored α or β look like? Denote such subgraph by $H_{\alpha,\beta}$. Explore the following coloring for inspiration.



Solution: Let $H_{\alpha,\beta}$ be the subgraph of G induced by edges colored by α or β. Notice that every vertex is incident to at most one edge colored α and at most one edge colored β. Hence $H_{\alpha,\beta}$ has maximum degree two. So it is a collection of paths and cycles. Also notice that the colors are alternating in these paths and cycles.

Theorem 2 (König)**.** *For every bipartite multigraph G, it holds*

$$\chi'(G) = \Delta(G).$$

Proof. Let $\Delta = \Delta(G)$. Suppose we have colored all the edges of G except edge $e = uv$. As there are at most $\Delta - 1$ colored edges at u, there must be a color i not present at u. Similarly, there exists a color j not used on the edges of v.

5: Look at the subgraphs induced by colors i and j and finish the proof.

Solution: In case that we can choose $i = j$ then we also color e by i and we are done. So assume that this is not possible, and so $i \neq j$.

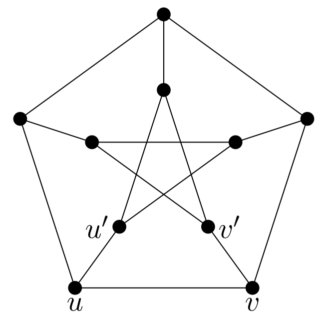
Now, consider the graph induced by edges of colors i and j, its components are paths and cycles, on whose edges these two colors alternating. The component that contains vertex u must be a path P as color j does not appear on the edges at v. The path P does not contain v, as then it must terminate at this vertex, and as i and j alternating on the edges of P, we infer that P is a a path of even length, but then $P + e$ is an odd cycle in G, a contradiction. So we can interchange the colors i and j on the edges of P, this way color i dismiss at u and it does not appear at v. But then color e by i to obtain a coloring of G.

□

One can give an alternative proof in the following way. As an exercise show that any bipartite graph is a subgraph of a bipartite regular graph. An easy consequence of the Hall theorem is that a regular bipartite (multi-)graph has 1-factor, in fact, it is a 1-factorable graph, i.e., there is a partition of its edges into 1-factors. And, these 1-factors induce an edge-coloring of the original graph.

1 Vizing's theorem

6: Show that the Petersen graph is not 3-edge colorable.



Hint: Suppose for contradiction that there is a 3-edge-coloring. If uv has color c, what colors are present at u' and v'? What is $\chi(C_5)' =$?.

Solution: Draw the Petersen graph in the usual drawing. Let uv be an edge of the outer 5-cycle C colored by c. Let C' be the inner cycle (usually not drawn in a planar way). Let u' and v' be the neighbors of u and v respectively that are not in C. Observe that each u' and v' are incident with an edge of C' colored by c. Since u' and v' are not adjacent, c must be on two edges of C'. The outer cycle C needs all three colors, hence C' must contain at least two edges of each of the the colors, which is a contradiction, since C' has just five edges.

But before we prove it, let us introduce a definition. Let G be a properly edge-colored graph and α, β two distinct colors used. Observe that the subgraph $H_{\alpha,\beta}$ of G induced by these two colors is comprised of even cycles and paths on which these two colors alternating. Notice that by swapping these two colors on a component of $H_{\alpha,\beta}$, the coloring still stays proper. Actually, we already use this technic in the above theorem. Subgraphs as $H_{\alpha,\beta}$ are called *Kempe chains*, as Kempe was the first to apply them in some arguments (though he did that for vertex colorings).

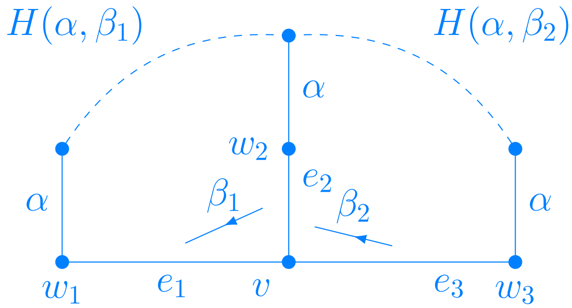
Theorem 3 (Vizing)**.** *Every simple graph satisfies*

$$\chi'(G) \leq \Delta(G) + 1.$$

Proof. Suppose this does not hold for G and let $\Delta = \Delta(G)$. Think of induction on the number of edges if you do not like contradiction with smallest counterexample. We may assume that we have colored all the edges of G but one $e_1 = vw_1$. Since we have $\Delta + 1$ available colors, there is a color missing at v, say α, and there is a color missing at w_1, say β_1. We may assume that we cannot choose α and β_1 to be the same color, since we could assign this color to vw_1 and get a contradiction.

7: Sketch the situation. What happens when you try to modify the coloring to assign β_1 to e_1? Hint - see $H_{\alpha,\beta}$ Are there conflicts? How to fix them?

Solution: Color β_1 must appear at v, say $e_2 = vw_2$ is colored by β_1. Move color β_1 from e_2 to e_1. We may assume that v, w_1, w_2 belong to the same $H(\alpha, \beta_1)$ component otherwise we can swap α and β_1 on this component and assign α on the edge e_2 to have a proper coloring of G.



Repeat this 'shift' over and over again. Means get colors $\beta_1 \neq \beta_2 \neq \beta_3, \ldots$.

8: Why can we take $\neq$ in the β_i and β_{i+1}?

Solution:

As Δ is finite, we encounter situation where edge $e_k = vw_k$ is uncolored and at w_k is missing some color β_k that satisfies one of the following:

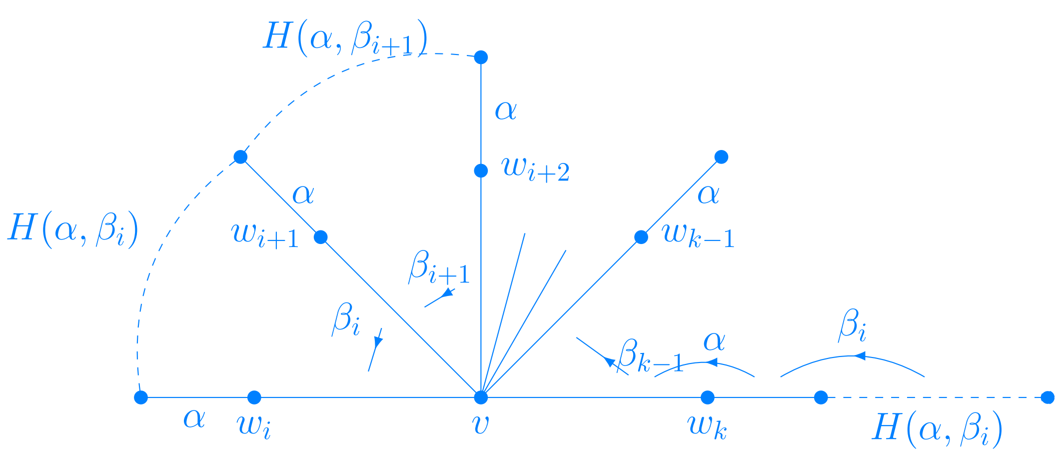
- β_k does not show at v, or
- $\beta_k = \beta_i$ for some $i < k-1$, i.e., a color that we encountered before.

9: Why will this happen?

Solution: If not before this will happen when $k = \Delta$, notice that then either β_Δ does not appear at v or it must be equal to some previous β_i (distinct from β_{k-1} as this color we had on vw_k).

10: How to finish the proof in either of the two cases?

Solution: In the first possibility we just put β_k on vw_k and we are done. And, in the second possibility, observe that v, w_i, and w_{i+1} belong to a component of $H(\alpha,\beta_i)$ in which w_k does not belong. So swap the colors α and β_i in the component of $H(\alpha,\beta_i)$ that contains w_k and at the end assign α to vw_k and we obtain a proper coloring of G.



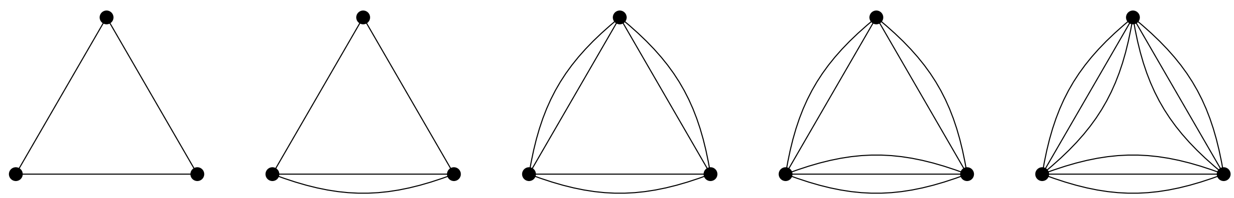
□

Vizing's theorem arises a very interesting problem. Let

- *Class I* be simple graphs G for which $\Delta(G) = \chi'(G)$,
- *Class II* be simple graphs G for which $\Delta(G) = \chi'(G) + 1$.

Thus, Q_3 is a Class I graph and Petersen is a Class II graph. We can ask for every graph is it in Class I or in Class II. From algorithmic point of view, it is NP-complete to decide for a graph which of these two classes is of Holyer. Also worthy to mention that Erdős and Wilson showed that almost all graphs are of Class I.

11: Show that Vizing's theorem does not hold for multigraphs. Consider the following graphs, called Shannon triangles.
Generalize the construction and find a constant c such that this construction is showing $\chi'(G) \geq c \cdot \Delta(G)$.



Solution: The line graphs are cliques. So the number of edges is the number if colors needed for edges. If each edge has multiplicity μ, then $\delta(G) = 2\mu$ and $\chi'(G) = 3\mu$. Hence $\chi'(G) \geq \frac{3}{2}\delta(G)$.

The following theorem gives an upper bound of χ' in term of Δ for multigraphs.

Theorem 4 (Shannon)**.** *Every graph G satisfies*

$$\chi'(G) \leq \left\lfloor \tfrac{3}{2}\Delta(G) \right\rfloor.$$

The *multiplicity* of a graph G, denoted by $\mu(G)$, is the maximum number of edges that are pairwise parallel, i.e., that have both end-vertices the same. Simple graphs have multiplicity 1. Vizing and Gupta independently generalized Theorem 3 to loopless multigraphs involving the multiplicity.

Theorem 5 (Vizing, Gupta)**.** *Every graph G satisfies*

$$\chi'(G) \leq \Delta(G) + \mu(G).$$

1.1 Goldberg conjecture

Comparing the last two theorems, Shannon bounds is sharper when $\mu > \Delta/2$, and oppositely for $\mu < \Delta/2$ sharper is the one of Vizing and Gupta. So, combining the last two results for multigraphs we have that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + \min\left\{\mu(G), \lfloor\frac{\Delta(G)}{2}\rfloor\right\}$$

telling us that we have many more possibilities than just two as it is for simple graphs. Let us state a well-known conjecture, which will somehow restrict the possibilities of chromatic index to just three possibilities.

12: Suppose we have some optimal edge coloring of G. Let S be a subset of vertices of G such that $|S| \geq 3$ is of odd order. Show that

$$\chi'(G) \geq \frac{2|E(G[S])|}{|S|-1}$$

Solution: Note that each color class uses at most $(|S|-1)/2$ edges, so we need at least $2|E(G[S])|/(|S|-1)$ colors to color $G[S]$.

Let

$$\rho(G) = \max\left\{\frac{2|E(G[S])|}{|S|-1} : S \subseteq V(G) \text{ with } S \text{ being odd and of size } \geq 3\right\}. \tag{1}$$

Obviously $\rho(G)$ is a lower bound for $\chi'(G)$. The next conjecture, proposed independently by Goldberg and Seymour is an attempt to preserve the dichotomy of simple graphs to only few case in multigraphs.

Conjecture 6 (Goldberg, Seymour)**.** *For every multigraph G, the chromatic index $\chi'(G)$ equals $\Delta(G)$ or $\Delta(G)+1$ or $\rho(G)$.*